Regarding Simple Harmonic Oscillation:

The PHASE CONSTANT (φ)

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Consider a particle undergoing *Simple Harmonic Oscillation*.
By definition, the particle moves smoothly back and forth along one dimension (or *axis*) of space – –
while (the) one dimension of time unfolds.

Let *y* stand for the axis along which the oscillation occurs.

This choice might suggest that we're picturing an axis aligned with gravity

(and thereby perpendicular to the horizon),

but it does not *need* to mean that.

***y* is simply 'the axis of oscillation' from here on in.**

At any given moment (in time, measured in, say, seconds)
the precise position (in space, measured in, say, meters)
of the particle
can be expressed by the differential equation below.

$$\frac{d^2y}{dt^2} = -\omega^2 y$$

In this function (directly above), ω is a constant.

Since the left side of the equation is necessarily measured in $\frac{\text{m}}{\text{s}^2}$,

so must be the right side. And *y*, of course, is measured in meters.

ω^2 is measured in $\frac{\text{radians}^2}{\text{s}^2}$.

A radian, strictly speaking,

is a ratio of $\frac{\text{(arc) length}}{\text{(radial) length}}$;

it is thus, in fact, dimensionless – –

like a percent, a fraction

or any other 'pure' coefficient for which all units have 'cancelled out'.

This can seem surprising or even wrong,

until/unless we come to regard the question "how large is that angle?"

as equivalent to asking "how much of an entire circle is that?"

($e.g.$: $\frac{\pi}{2}$ radians = 90° = $\frac{1}{4}$ circle).

We do not regard '*circle*' as a standarized unit of measurement:

Any two meters or any two seconds you pick,

will be, by definition, the same size as each other.

but any two circles you pick are probably NOT the same size.

Why should we expect that to change – –

merely because we decide to sub – divide the circle into 360 or ~6.3 (2π) parts?!

To 'solve' the above Dif. Eq. is to find some expression which relates the dependent variable (here, y) to the independent variable (t):

$$y = f(t)$$

AND for which the statement about derivatives is true.

Early on, we found a solution and have it used it ever since:

$$y = A\cos\omega t$$

(*A*, ω constants).

The above is easily 'proved' (checked) to be a solution, but we never proved or even claimed that it was the ONLY solution: It isn't. Consider:

$$y = A\cos(\omega t + \varphi)$$

(*A*, ω AND φ constants).

It seems that anything in our minds which helped lead us to guess the first solution (graphs, units, limiting cases, etc.) would probably apply to the above.

Check:

$$y = A\cos(\omega t + \varphi)$$
$$\frac{dy}{dt} = -A\omega\sin(\omega t + \varphi)$$

Given harmonic oscillation and its characteristic differential equation,
we are concerned with the SECOND derivative.
So let's continue.
It is quite worthwhile to pause and think once we find ... :

$$\frac{d^2y}{dt^2} = -A\omega^2 \cos(\omega t + \varphi).$$

since $y = A\cos(\omega t + \varphi),$
then re − arrange for convenience:

$$\frac{d^2y}{dt^2} = -\omega^2 A\cos(\omega t + \varphi)$$

and find

$$\frac{d^2y}{dt^2} = -\omega^2 y$$

!!!

That is, this 'new' (or at least different − looking) function is
'*another*' solution to the differential equation for simple harmonic oscillation!

NOTE: IFF you are willing to view everything inside the parenthesis
as one chunky variable, then this result starts to feel like
the same derivative as for $y = A\cos(\omega t)$**.**
It's not literally the same, but it's similar
in an important way − −for a reason that's not a coincidence.

Because φ is a *CONSTANT*;
the chain rule demands that we multiply the derivative of the outer function

$$\frac{d}{dt}(\cos(blahblah)) = -\sin(blahblah)$$

by the derivative of the inner function:

$$\frac{d}{dt}(blahblah \cdot t) = blahblah$$

so we got:

$$-blahblah \sin(blahblah \cdot t) =$$
$$-A\omega \sin(\omega t)$$

Now, here, with this new possible solution, the only things that's changed is the inner function.
The inner function does indeed appear to have changed,
but not by a way that affects its derivative.
The complete (1st) derivative, therefore, stands intact – –
as must every subsequent derivative of that.

$$\frac{d}{dt}(\omega t) = \frac{d}{dt}(\omega t + 0) = \frac{d}{dt}(\omega t + 1) = \frac{d}{dt}(\omega t + 2) = ...$$
$$\frac{d}{dt}(\omega t + 0) = \frac{d}{dt}(\omega t + \varphi)$$

So:

Another way to see why

$$\frac{d}{dt}[A\cos(\omega t+\varphi)]=\frac{d}{dt}[A\cos(\omega t)]=-A\omega\sin(\omega t)$$

is to say that we HAVE BEEN working with the longer function the whole time, but only for the clear – cut case in which $\varphi = 0$.

From that point of view, we are now simply raising the concern:
What happens if we release the unstated restriction (that $\varphi = 0$)?
If we free ourselves from the constraint that $\varphi = 0$,
then, for any given A and ω, we see a whole set of solutions

$$(\ldots y=A\cos(\omega t-2), y=A\cos(\omega t-1), y=A\cos(\omega t),$$
$$y=A\cos(\omega t+1), y=A\cos(\omega t+2)\ldots)$$

where we used to see just one ($y=A\cos(\omega t)$).

So what does all this mean?
Does this increase our understanding or does this increase our confusion?

Let's directly confront the new and confusing piece:
WHAT, specificially, DOES φ MEAN?
What is the difference, for example, between:

$y = 2\cos(\pi t)$ and $y = 2\cos(\pi t + 3)$

(for which 2, π, 3 are arbitrary examples of constants)?

In physics, we tend to decipher functions and resolve matters of this kind by asking:

To what ***PHYSICAL MEASUREMENT*** does φ refer?

First, look at units.
It *must be* measured in the same units as ωt; therefore,

φ is measured in radians.

so it seems that φ measures some kind of 'angle' – –
but only to the extent that ωt does:

Whenever considering periodic motions such as oscillations and waves, it becomes extremely helpful to maintain our fundamental understanding (definition) of angle (measured in radians).

If s stands for '*arc length*' along some section of a circle [meters]
and r stands for the radius of that circle [meters], then:

$$\theta \equiv \frac{s}{r}$$

*AN ANGLE is ('simply') A *FRACTION* OF A CIRCLE.

When we are willing to be slightly casual,
'circular' describes more than strict geometric objects:
We consider sequences, processes, rhythms, etc., to be circular
as long as they (identically & measurably)
repeat themselves with no forseeable ending:
as long as there is periodicity.

So, in the context of oscillation:
ωt, φ and other related radian terms
refer to portions of the oscillation cycle,
fractions of the *period*:

An angle that takes you exactly once around a circle measures 2π.
So, 2π means 1 full cycle: 'there and back'

(e.g.: A to $-$ A then back to A

OR $\frac{A}{2}$ to $-$ A and back to $\frac{A}{2}$ via side $-$ trip from $-$ A through $-\frac{A}{2}$

OR 0 to 0 via side trip from 0 through $-$ A, etc...)

π means half a cycle: 'there', ..,

... $\frac{\pi}{6}$ means 'a twelfth' of a cycle (a sixth of the way 'there') ... etc.

So, φ refers to a delay or head – start in the oscillation cycle.

A delay or head – start in the oscillation cycle is generally referred to as a '***Phase Shift***';
φ is generally referred to as the '***Phase Constant***'.

To make discussion of phase as clear as possible,
we will choose to assume we will always be comparing oscillators to
the 'control' or 'unshifted' (simplest possible) case of $\varphi = 0$.
With respect to this 'unshifted case', we will choose to assume that
the oscillator 'begins from rest'.
This simply means that we choose to start our timers
once the oscillator has reached a turning point.
We could start anywhere and we have to start somewhere,
so this choice makes our life more convenient without limiting our knowledge.

Thus,

(assuming $v_0 = 0$ for $\varphi = 0$):

When $\varphi = 0$, then $y = A$ at $t = 0$.

But if, for example, $\varphi = -\frac{\pi}{4}$,

then that means we have to wait for an eighth of the period to elapse until
we find the oscillator at the Amplitude:
It does NOT begin at maximum displacement from equilibrium.