Practice Exam 1 SOLUTIONS, Part 1

A rogue planet called Melancholia is found to have a uniform density. It's mass, M_M, is approximately $7\times10^{26}\ kg$ and its radius, R_M, is approximately $2\times10^{7}\ m$.

A tunnel runs from a point on the surface of Melancholia, through the center of the planet, all the way out to a point on the opposite side. Frictional forces inside the tunnel is negligible.

A mass of 50 kg is placed gently in the center of this tunnel, at the very center of Melancholia.

a. What is the weight of this mass (force of gravity on it) at this point?

Weight = 0
At center of planet $F_g = 0$, bec. object is equally surrounded by planet's mass on all sides.

b. An incredibly long cable is used to pull the object away from the center. What happens to its weight as it rises towards the surface? What is its weight when it reaches the surface?

As the object rises, its weight increases (bec. more of the planet is on one side of it than on the other.)
When it reaches the surface, it is a distance of R (=$2\text{x}10^{7}$ m) from the center & all the weight is "below" it:

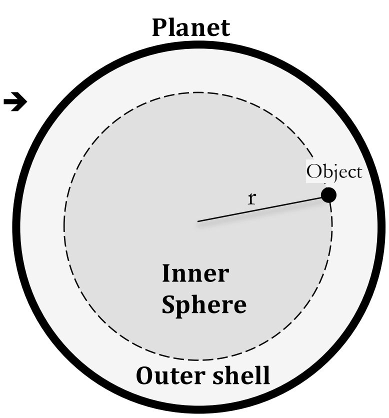
Weight at surface = F_G at surface = $\frac{GMm}{r^2} = \frac{(6.67\times10^{-11})(7\times10^{26})(50)}{(2\times10^{7})^2}$ = 5836 n.

c. And now the kicker: the object is lifted all the way to the surface and then dropped back into the tunnel. ***Given only Newton's 2nd Law of Motion, Newton's Shell Theorem, and Newton's Universal Law of Gravitation, <u>PROVE that the motion of this object will be SHO</u>***.

To do this, you will first need to derive (prove) an expression for the force of gravity on the object as it falls, in terms of its distance from the center of the planet, r.

Show & justify all steps.

Along the way, you will have to:
1. Use diagrams and words to explain how the planet is divided into two regions;
2. Explain what sort of gravitational force (if any) each region exerts on the object;
3. Derive an equation for the mass of the inner region;
4. Derive an equation for the force of gravity on the object as a function of its distance from the center of the planet;
5. Produce a differential equation & show that the motion is SHO.

CLAIMS	JUSTIFICATIONS
1. Consider a moment when the object is partway down: some distance ***r*** from the center.	1. Not a claim.
2. Imagine the planet divided into two regions: all the matter at a distance < ***r*** from the center forms a SPHERE. All the matter at a distance > ***r*** from the center forms a SHELL.	2. See diagram ➔ 

	The diagram is fine, but for those who care, here is a more rigorous justification for claim 2: "The set of all points less than a given distance r in 3D space" is the definition of a sphere. If you cut a small sphere out of a larger one, you get a shell.
3. The SPHERE has radius ***r***.	3. See diagram fr step 3.
4. Our object is INSIDE of the SHELL but it is OUTSIDE of the sphere.	4. See diagram fr step 3.
5. Our object is INSIDE of the SHELL but it is OUTSIDE of the sphere.	5. See diagram fr step 3.
6. The net gravitational force from the OUTER SHELL is zero.	6. **Wait—why are all these justifications left blank??? These steps DO need justifications. YOU have to come up with those justifications in YOUR words.**
7. The net gravitational force from the INNER SPHERE can be calculated by treating the sphere as an ordinary point-mass located at its center.	7.
8. Net grav force on the object = net grav force from the inner sphere = $F_{G.IS}$.	9. Follows from steps 7 & 8.
9. $F_G = -\frac{GMm}{r^2}\hat{r}$	9.
10. $\vec{F}_G = -G\frac{(M_{IS})m}{r^2}\hat{r}$	10.
11. We will now derive the M_{IS}, the mass of the Inside Sphere.	11. Not a claim: no need for justification.
12. The density of melancholia is equal everywhere.	12.
13. The average density of the whole planet = the average density of any region inside of it.	13.
14. ρ_M (density of Melancholia) $= \rho_{IS}$ (density of Inner Sphere)	14.
15. $\rho_M = \frac{M_M}{V_M}$; $\rho_{IS} = \frac{M_{IS}}{V_{IS}}$	15. (Note: bec these two statements have the same justification, we have put them on the same line.)
16. $V_M = \frac{4}{3}\pi R^3$; $V_{IS} = \frac{4}{3}\pi r^3$	16. (Again: same justification, same line.)
17. $\frac{M_M}{\frac{4}{3}\pi R^3} = \frac{M_{IS}}{\frac{4}{3}\pi r^3}$	17.
18. $\frac{M_M}{R^3} = \frac{M_{IS}}{r^3}$	18. Algebra. (You may skip this step if you like)
19. $M_{IS} = \frac{(M_M)r^3}{R^3}$	19. Algebra.

20. $\vec{F}_G = -G\frac{\left(\frac{(M_M)r^3}{R^3}\right)m}{r^2}\hat{r} = -\frac{GM_Mm}{R^3}(r)\hat{r}$	20. Substitution: step 20 into step 11.
21. $(r)\hat{r} = \vec{r}$	21. Definition of $\hat{r}$.
22. $\vec{F}_G = -\frac{GM_Mm}{R^3}\vec{r}$	22.
23. The above function holds at any distance ***r*** from the center of the planet, so long as we are inside the planet, i.e. $r < R_M$. Therefore, as it falls, it is constantly subject to a gravitational force given by that function.	23. Follows from everything done so far: no assumptions were made that restrict ***r***.
24. No other forces act on the object as it falls through the tunnel.	24.
25. $\sum \vec{F} = m\vec{a}$	25.
26. $-\frac{GM_Mm}{R^3}\vec{r} = m\vec{a}_{object}$	26.
27. $-\left(\frac{GM_M}{R^3}\right)\vec{r} = \vec{a}_{object}$	27.
28. $-\left(\frac{GM_M}{R^3}\right)\vec{r} = \frac{d^2x}{dt^2}$	28.
29. $-\left(\frac{GM_M}{R^3}\right)\vec{r} = \frac{d^2r}{dt^2}$	29. ***r*** = dist. fr. center of planet. If we define x = 0 to be the center of the planet, then $\vec{x}$ is equivalent to $\vec{r}$.
30. G, M_M, and R are all constants. Only ***r*** varies as the object falls.	30.
31. The object's acceleration is negatively proportional to its position. OR just: $\boldsymbol{a} \propto -\boldsymbol{x}$	31.
32. The object's motion is an instance of SHO. **Q.E.D.**	32. Step 31 is the definition of simple harmonic motion.

Again, all these blanks are here because YOU have to come up with YOUR own justifications.

These steps DO ALL need justifications.

d. Based on your solution to C, find the length of time required for the object to reach the center of Melancholia.

$$\omega = \sqrt{\frac{GM_M}{R^3}} = \sqrt{\frac{(6.67\times10^{-11})(7\times10^{26})}{(2\times10^7)^3}} \approx 0.0024$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{0.0024} \approx 2618\ s$$

Time to center = one quarter period = $\frac{2618}{4} \approx 654.5$ seconds ≈ 10.9 minutes

e. Where is the object's PE highest? Where is it lowest? How does E_{total} change over time? Explain.

PE is highest when the object is at the extremes of its oscillation, near the surface of the planet, because at this time it is farthest from equilibrium, so gravity can do the maximum positive work on it as it returns to equilibrium. PE is lowest when the object is at the center of the planet, because this is the equilibrium position, where all forces are balanced. Energy goes back and forth between PE and KE but total energy remains constant over time. This is because friction is negligible and gravity is a conservative force.

f. The same object is caught again, lowered to a point just 100 m above the center of the planet and dropped. What will be different about its motion? What will not be different? How long will it take it to fall the 100 m to the very center?

The AMPLITUDE will change (from $2x10^7$ m to 100 m), but frequency and period will remain the same, because, in SHO, frequency & period are amplitude-independent.

It will still take 654.5 seconds to reach the center.

Another planet called Alegria is found to have a density ***that increases as you approach the center of the planet***.

A tunnel is dug from a point on the surface of Alegria, through the center of the planet, to a point on the surface on the opposite side.

An object is dropped into this tunnel. Assume that friction inside the tunnel is negligible.

g. As the object falls towards the center of Alegria, does its speed increase, decrease, or remain the same? Explain.

Direction of force is towards center of planet, so direction of ***a*** is towards center. Motion is also towards center. ***a*** in the same dir. as ***v***: speed <u>increases</u>.

h. After it passes the center, does its speed increase, decrease, or remain the same? Explain.

Direction of force is still towards center of planet, but now motion is away from center. ***a*** in opp. dir. from ***v***: speed <u>decreases</u> until the object stops & turns around.

i. Will the object oscillate? Explain your reasoning in English, not math.

Yes. Acceleration is always back towards the center (i.e. opposite displacement from center). It will keep being pulled back, but gravity is a conservative force, so the energy will keep being converted back and forth, kinetic, potential, kinetic, potential, and never stop.

j. Now the kicker: ***is the object's motion an instance of SHM.***

Provide a compelling argument for your answer. This will ***not*** be a mathematical proof. It will almost certainly involve some words, possibly also a couple equations. It will likely make use of the definition of SHO.

<u>**NO**</u>**.** On Melancholia the motion was SHO, bec. $F_{G.IS}$ was a linear function of distance from center, and therefore ***a*** was proportional to the neg. of position (definition of SHO). But this resulted from the condition of <u>uniform density</u>. If density gets higher towards the center, then F_G will not decrease at the same rate—or may not decrease at all. $F_{G.IS}$ will therefore not be linear with respect to ***r***, so ***a*** will not be proportional to ***r***. Therefore, the motion will not fit the definition of SHO.