Unthung:
Building the Wave Equation From Empirical Data

E & M: A Saint Ann's Study of I & P.

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A) READING: *The experimental background.*

Two students stand meters away from each other in an ideal physics lab (one in which budgetary, architectural and temporal constraints are negligible). They hold the opposite ends of a long, narrow and tightly coiled 'slinky'-like spring. They hold their ends firmly—so that tension keeps the spring in a pure and stable horizontal orientation for significant time. After a reliably steady state is achieved, one student begins shaking her end up and down in small, controlled disturbances (and the other student stays still). Once the small disturbances themselves have fallen into a regular rhythm, stopwatches and video cameras (a.k.a.: 'phones') start rolling. An experiment is underway; a point designated t = 0 sec is passed… and past…

One convenient way to picture the situation is as follows:

https://www.youtube.com/watch?v=y66PSaiGH7Y&list=PLY12vIxNLhy5pUg2E1EEZ4jHs8Ulxxf4a&index=3

Some might prefer to picture the situation more like this:

https://www.youtube.com/watch?v=F9tNlTLh7ts.

The two models might appear quite distinct, but they both capture all the essential features for this context.

After some kind of video resembling something like one of the above is captured, video-playing/freeze-frame/editing software is deployed– along with internal rulers, scaling/conversion functions and similarly appropriate measurement tools. (Remember, this is an idealized laboratory, of a caliber and quality you are likely to find only in your imagination—particularly if your imagination happens to escort you toward some urban public university and into, say, its college of criminal justice.)

B) READING: *Data from one fixed horizontal position.*

The students choose an origin and impose an X-Y coordinate system onto the video (with, for example, a transparent overlay of graph paper). The X-axis lies along the original equilibrium orientation of the spring. The Y-axis, of course, is perpendicular to that; in the case of this experiment, as with many, the Y-axis is parallel to the direction of gravity.

The students then select one specific and convenient $\boldsymbol{x}$ coordinate. For this procedure, they decide to focus their attention only and always at this particular $\boldsymbol{x}$ displacement from the origin; perhaps it is a comparatively vivid place such as where the background wall has discoloration or a thumbtack or the like. They play the video frame by frame and, by inserting little digital mark as measurements, track the $\boldsymbol{y}$ (vertical) coordinate of the spring at that fixed $\boldsymbol{x}$ (horizontal) place. At that $\boldsymbol{x}$ coordinate, as at all $\boldsymbol{x}$ coordinates, they notice the spring going up and down continually. For each frame, they note the new $\boldsymbol{y}$ position of the spring. They know the '*frame-rate*' at which their device captured video. They therefore know the amount of time between each pair of successive frames. So each $\boldsymbol{y}$ position is easily mapped to the total amount of time elapsed since the beginning of the experiment.

All the above-described measurements are easily expressed as ordered pairs of the two quantities that vary (vertical position, time) while one quantity (horizontal position) is held constant. The $(\boldsymbol{y}, \boldsymbol{t})$ data are thus turned into a scatter-plot. A simple, smooth pattern is apparent among the points; a trend line is drawn. A simple function, $\boldsymbol{y} = \mathrm{f}(\boldsymbol{t})$, that reasonably describes the shape of this trend line is as follows:

$$y = A_1 \cos(\omega t + \varphi_1).$$

Feeling that this function is somehow familiar and therefore possibly important, they put the graph and implied relationship aside—in an extremely safe and cozy place.

C) READING: *Data from one fixed time*.

The students then select just one vivid frame from the entire video. They make little digital measurements of the spring as it appears in this frame. Here in one frame, time no longer varies. The spring can be seen stretching out and inhabiting space in both dimensions: X and Y. Although quite flexible and distortable, the spring itself remains a 1-dimensional creature; it can bend from a height-wise to a width-wise orientation, but it does not spontaneously gain width or thickness. At any and every given horizontal position (from which there are now many to choose), the spring therefore occupies one and only one vertical position. Digital measuring marks are inserted along the spring's body and the X-Y coordinate system is noted.

These marks allow for a new set of ordered pairs—all of which apply at one particular moment. The $(\boldsymbol{y}, \boldsymbol{x})$ pairs indicate the dependence of the spring's vertical position on horizontal position, for one fixed (constant) value for $\boldsymbol{t}$.

The measurements from this one frame fall almost immediately into their own graph: a graph of $\boldsymbol{y}$ as a function of $\boldsymbol{x}$. The graph's trend line is evident. The shape of this trend line is described by the following function:

$$y = A_2 \cos(kx + \varphi_2).$$

D) <u>QUESTIONS (for you to answer—*on a clean, separate sheet of paper*):</u>

1. In what units must $\boldsymbol{\omega}$ be measured?

2. In at least one complete sentence of English, describe the particular aspect or feature of the spring's physical motion that is expressed by $\boldsymbol{\omega}$.

3. In what units must $\boldsymbol{k}$ be measured?

4. In at least one complete sentence of English, describe the particular aspect or feature of the spring's physical motion that is expressed by $\boldsymbol{k}$.

5. Assuming the principle of superposition applies, propose one simple expression that can accurately and generally capture the vertical position of the spring, $\boldsymbol{y}$, as a simultaneous function of two independent variables: horizontal position, $\boldsymbol{x}$, <u>and</u> time, $\boldsymbol{t}$. If you wish, you may assume that $\varphi_2 - \varphi_1 = 0$.

6. In at least two complete sentences of English, explain/justify why the function you proposed above is reasonable: That is, why do you believe that it conveys all the same information as the two separate functions taken independently? How did you come up with it?

7. Recalling the procedures used to collect the first set of data, temporarily pretend that the $\boldsymbol{x}$ variable in your function is, in fact, held at one fixed value. That is, temporarily and 'for the sake of argument', treat $\boldsymbol{x}$ as a constant. Given this temporary assumption, differentiate your function twice. Your result is known as second ***partial derivative*** of $\boldsymbol{y}$ as a function of $\boldsymbol{t}$ (where $\boldsymbol{x}$, as stated above, is treated as a constant).

8. You might well guess what to do next. Mathematically capture the second procedure that was followed for data collection: Assume for the moment that time, $\boldsymbol{t}$, is CONSTANT. That is, imagine that you are focusing your attention on one frame or snapshot of the wave motion. Given this temporary assumption, differentiate your function twice—in order to get the second ***partial derivative*** of $\boldsymbol{y}$ as a function of $\boldsymbol{x}$ (where $\boldsymbol{t}$, as stated above, is treated as a constant.)

9. Now find a relation between the two derivatives you computed above. This relation will be a second order differential equation. Iff you have done any reading whatsoever (in, for example, your textbook or on, for example, the web), then this second-order differential equation relation should look extremely familiar! Hint: It will be an ***equation*** that applies to ***waves***.

10. Given the equation you derived in (9), above, determine the relationship (equation) between ***wave speed***, ***angular frequency*** and ***angular wavenumber***.

11. Given the relation (equation), you derived in (10), above, determine a relationship between ***wave speed***, ***standard frequency*** and ***wavelength.***

    The three relations you found above, in parts (9), (10) and (11), are true for all time—in life and in this class.

12. A CENTRAL POINT: In at least three complete sentences of English, and in light of all the above,

    Describe how you might build (construct, create, guarantee) wave motion if all you had as raw ingredients were a huge collection of simple harmonic oscillators.

13. In at least one complete and original sentence of your own chosen English words,

What IS wave motion?

(Consider: In what sense is it legitimately describable as motion? In what sense is it *not* legitimately describable as particle motion?) For this question, you are being asked to propose an original definition, not merely identify an interesting property or two.